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Manuscript received March 2, 1964; revision received June 29, 1964; paper accepted June 29, 1964.

Stability Criteria for Miscible Displacement of Fluids from a Porous Medium

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Stability criteria are formulated for displacement of a fluid from an infinite porous medium by a second more dense fluid miscible with the fluid being displaced. Pure displacing fluid is separated from pure displaced fluid by a region of constant thickness in which the two fluids interpenetrate. The steady displacement process takes place at a constant velocity in the direction of gravity. It is shown that appreciable differences in pure-component densities, viscosity variation within the zone of interpenetration, and the displacement velocity can have a major effect on the stability criteria of such a flow.

Persons concerned with the production of petroleum crude from underground reservoirs have in recent years become greatly interested in so-called *secondary recovery processes.* Among these processes are techniques whereby the crude is displaced by a nonaqueous fluid (for example, liquefied petroleum gas) which is miscible with the crude. By driving the solvent fluid with a third relatively inexpensive material, it was hoped that the solvent would essentially dry clean the reservoir with the resultant recovery of virtually all of the oil in place plus the injected solvent. Such a scheme potentially offers great improvement over customary recovery techniques which obtain something less than 50% of the oil in place.

Ideally one envisions a pistonlike displacement of the crude by the solvent with some interpenetration of the two materials owing to diffusion. In practice this happy result has seldom been achieved. Experimenters have noted that the solvent fluid, which is generally less viscous than the crude, forms protrusions into the oil resulting in an ill-defined boundary between oil and solvent. The result is that fingers of solvent reach the producing well long before much of the crude has been displaced. Hence, the fraction of oil recovered at the moment of solvent breakthrough is disappointingly low.

It is well known that the formation of fingers is a result of hydrodynamic instability of the displacement process. Existing attempts to elucidate the nature of the instability have either been concerned with experimental or computational studies of finger growth and its effect on oil recovery (*2, 8*) or have consisted of mathematical analyses which were idealized to the extent that some essential features affecting the instability were not included.

The present work does not pretend to analyze a real reservoir problem. However, it does offer, for the first time to the author's knowledge, an analysis in which the relevance of various dimensionless parameters to the displacement process is made apparent.

Although primary interest in miscible displacement has come from the petroleum industry, the general hydrodynamic problem is one which is pertinent to many facets of chemical engineering. Hence, this paper is presented as an attack on a problem of general hydrodynamic interest.

THEORY

Previous Work

The problem being considered is related to the classic Rayleigh stability problem of a fluid heated from below (*4*). In one version of that problem the fluid is contained between two stationary horizontal planes, the bottom plane being maintained at a constant temperature greater than the constant temperature of the top plane. Because of the adverse density gradient maintained by the temperature gradient, gravitational forces tend to cause convective motion of the fluid. However, this tendency is countered by thermal conductivity of the fluid, which allows for heat transport by conduction, and by the fluid viscosity, which is related to the drag force opposing convective motion. The relative magnitude of the forces aiding convection with respect to those tending to suppress it is embodied in the dimensionless Rayleigh number

$$R = -\frac{g\,d^4}{\kappa_T \mu}\,\frac{d\bar{\rho}}{d\,x}$$

For the geometry cited above one finds that convective motion may occur for $R > 1708$, but for lower values of the Rayleigh number the fluid is stable to small perturbations in temperature, and the mode of heat transport is purely conductive.

In 1948 Lapwood (*7*) studied the analogous thermal convection problem for a fluid in a porous medium. Using the usual technique of introducing small perturbations into a steady state solution, he linearized the differential equations of momentum and heat transport and solved the resulting equations for a number of different boundary conditions. A similar problem involving transport of mass rather than transport of heat was solved in the first of a series of papers by Wooding (*14*). He considered a long vertical tube filled with porous material and containing a viscous fluid which, at steady state, exhibited a small linear variation in density from one boundary to the other. The fluid viscosity was assumed to be a slowly varying function of density. For this case the appropriate analogue to the Rayleigh number is the dimensionless group

$$\frac{-k g\, d^2}{\kappa\, \epsilon\, \mu}\left(\frac{d\bar{\rho}}{d\,x}\right) \tag{1}$$

In a later paper (*15*) Wooding discussed the correspondence which exists between Hele-Shaw flow and flow in a porous medium. This observation, first noted by Saffman and Taylor (*12*) in a study of immiscible displacement, led to stability criteria for flow through a porous medium contained between two vertical parallel planes. As in previous studies, it was assumed that fluid density was a linear function of height and that changes of viscosity and diffusivity with density were of second order. The appropriate dimensionless group which provides a criterion for stability is similar to that given above. Rachford (*11*) has recently shown that the analogy between Hele-Shaw flow and flow in a porous medium has significant limitations.

A related problem is Rayleigh instability of a heated liquid rising slowly through a semi-infinite porous medium toward a surface containing a pool of cool liquid (*16*). Wooding's treatment of this model, which approximates certain geophysical situations, was the first to consider a nonlinear steady state density distribution along with first-order viscosity variation. The viscosity was assumed to be a linear function of the density.

Two papers have dealt with stability characteristics of an unsteady flow. Wooding (*17*) has considered a vertically upward displacement proceeding from an initial state in which a sharp interface exists between two miscible fluids, each semi-infinite in extent, with densities $(\rho_o + \Delta)$ and $(\rho_o - \Delta)$ above and below the interface, respectively. It was assumed that $\Delta << \rho_o$ and that, to first order, viscosity was a linear function of density. A second type of unsteady state flow has been studied by Perrine (*10*). The stable motion consists of displacement from a two dimensional rectangular reservoir of finite height and length and oriented at a known angle to the direction of gravity. The basic differential equations require the difference of density between displaced and displacing fluids to be small with respect to pure-component densities. Viscosity was assumed to be an exponential function of density. A Fourier component of the perturbation which is superimposed upon the stable solution in order to determine whether the perturbation will decay or grow with time, indicating stability or instability, respectively, is expressed by

$$\eta = \tilde{\eta} \exp\,[i\,f(x, y, z, t)]$$

where η is concentration, velocity, or pressure, and $\tilde{\eta}$ is a constant. An initial condition imposed upon f is

$$f(x, y, z, 0) = \gamma x + \delta y + n\pi z$$

where γ and δ are constants and n is an integer. The constant pre-exponential factor requires that an initial perturbation, say in concentration, not be attenuated throughout the length of the reservoir.

Since the systems described above are composed of miscible components, effects due to interfacial tension have not been considered. Surface effects are also postulated to be of secondary importance in the present paper. However, it has been shown [see, for example, Pearson (*9*)] that surface effects can play a dominant role in systems with a single liquid phase.

Present Work

The treatment presented here is the first to include the effect of significant differences in density between the two pure components. Though the assumption of small density differences is usually fulfilled in problems of thermal instability, composition-induced density gradients are less likely to satisfy the condition $(\rho_{AP} - \rho_{BP}) << \frac{1}{2} (\rho_{AP} + \rho_{BP})$. It is also unrealistic to neglect the effect of viscosity change with concentration or to assume a linear dependence of viscosity on concentration. The present work assumes an exponential relation between viscosity and concentration. It shall be seen that inclusion of these two complications, along with a finite velocity of displacement, has a major effect upon the stability limits of the problem.

For a binary system the description of motion through a porous medium is contained in the continuity equation, the equation of motion, and a diffusion equation (continuity equation for a single species). In writing these equations one must be careful to distinguish between different definitions of velocity. Since one is seldom concerned explicitly with the point velocity of a fluid flowing through a porous medium, it is convenient to define a superficial or filter velocity **q** from the volumetric flow rate Q_v of fluid through a cross section A having a unit normal vector **n** and containing both pores and solid particles. The orientation of A is taken in the direction of the mean velocity vector of the fluid over the region under consideration. Then

$$\mathbf{q} = \frac{Q_v}{A} \mathbf{n} \tag{2}$$

From this one may define a pore velocity

$$\mathbf{v} = \frac{\mathbf{q}}{\epsilon} \tag{3}$$

The continuity equation for the fluid mixture may be written

$$\frac{\partial \rho}{\partial t'} + \nabla' \cdot (\rho \mathbf{v}) = 0 \tag{4}$$

The proper form for an unsteady state momentum equation employing **q** or **v** is not obvious. Strictly speaking, Darcy's law is applicable for steady flow under conditions such that inertial effects are negligible. However, one wishes to describe a flow in which the substantial derivative of **q** is not zero. It shall be assumed that under such conditions the viscous drag on the fluid can still be approximated by Darcy's law. One then obtains an equation of motion similar to that used by Wooding (*14*):

$$\mathbf{q} + \frac{\rho k}{\mu \epsilon} \frac{D\mathbf{q}}{Dt'} = -\frac{k}{\mu}(\nabla' p - \rho \mathbf{g}) \tag{5}$$

For a binary system $A + B$ the continuity equation for component A is written as

$$\frac{\partial \rho_A}{\partial t'} + \nabla' \cdot (\rho_A \mathbf{v}_A) = 0 \tag{6}$$

where, by definition

$$\rho \mathbf{v} = \rho_A \mathbf{v}_A + \rho_B \mathbf{v}_B \tag{7}$$

$$\rho = \rho_A + \rho_B \tag{8}$$

The usual binary diffusion coefficient is defined in terms of a mass flux of one component; for example, $\mathbf{j}_A$, with respect to the mass average velocity (*1*). Then

$$\mathbf{j}_A = \rho_A(\mathbf{v}_A - \mathbf{v}) = -\rho \kappa \nabla' \omega_A \tag{9}$$

where

$$\omega_A = \frac{\rho_A}{\rho_A + \rho_B}$$

Combination of Equations (5) through (8) leads to a diffusion equation for component A:

$$\frac{D\rho_A}{Dt'} + \rho_A \nabla' \cdot \mathbf{v} = \kappa \nabla' \cdot [\nabla \rho_A - \rho_A \nabla' \ln \rho)] \tag{10}$$

The last term has not been included in previous work and is important when the two species have pure-component densities which are appreciably different from each other.

Since average pore velocities $\mathbf{v}_A$ and $\mathbf{v}$ were used in Equation (9), it is not necessarily a correct description of the flow over a macroscopically small but microscopically large element of volume. Velocity gradients perpendicular to the direction of flow, which exist within the element of volume, can cause dispersion effects described by Taylor (*13*). This complication can frequently be handled by defining κ to be an effective dispersion coefficient rather than the molecular diffusivity. In general, the dispersion coefficient is dependent upon both velocity and direction. However, in the present work κ was assumed to be a scalar constant. It has been shown (*2, 3*) that under most conditions of reservoir displacement one may consider κ to be a constant of the same order of magnitude as the molecular diffusion coefficient and to be independent of both velocity and direction.

Equations (4), (5), and (10) are the basic differential equations which describe the system. They can be rewritten in terms of the variables **v**, ρ_A, and p. It will also be convenient to write these equations with respect to a coordinate system moving in the x direction at a velocity v_0. Then, setting

$$x = x' - v_0 t \quad z = z'$$
$$y = y' \quad t = t'$$

one obtains

$$\frac{\partial \rho_A}{\partial t} - \mathbf{v}_0 \cdot \nabla \rho_A + B \nabla \cdot \mathbf{v} + \nabla \cdot (\rho_A \mathbf{v}) = 0 \tag{11}$$

$$\frac{\epsilon \mathbf{v}}{\rho_{B_p}} + \left[1 + \frac{\rho_A}{B}\right] \frac{k}{\mu} \left[\frac{D\mathbf{v}}{Dt} - \mathbf{v}_0 \cdot \nabla \mathbf{v}\right] = -\frac{k}{\mu}\left[\frac{1}{\rho_{B_p}} \nabla p - \mathbf{g}\left(1 + \frac{\rho_A}{B}\right)\right] \tag{12}$$

$$\frac{\partial \rho_A}{\partial t} - \mathbf{v}_0 \cdot \nabla \rho_A + \nabla \cdot (\rho_A \mathbf{v}) = \frac{\kappa B}{B + \rho_A}\left[\nabla^2 \rho_A - \frac{1}{B + \rho_A} (\nabla \rho_A)^2\right] \tag{13}$$

where the relation between pure-component and mixture densities

$$\rho = \rho_{B_p}\left[1 + \frac{\rho_A}{B}\right] \tag{13a}$$

has been used.

One is interested in the behavior of small perturbations superimposed upon steady state solutions to Equations (11) through (13). Thus one writes

$$\rho_A = \overline{\rho}_A + \rho_{A_1} \tag{14}$$

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}_1 \tag{15}$$

$$p = \overline{p} + p_1 \tag{16}$$

$$\mu = \overline{\mu} + \mu_1 \tag{17}$$

Since one is interested in describing behavior of small perturbations, one can substitute Equations (14) through (17) into Equations (11) through (13), linearize the results by dropping all terms containing products of perturbations, and subtract the steady state equations. The re-

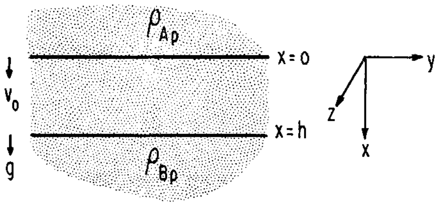

Fig. 1. Geometry of flow system.

sults are presented below in component form for a system where $\bar{\mathbf{v}} = \mathbf{v}_0 = \text{const.}$:

$$\frac{\partial \rho_{A_1}}{\partial t} + B\,\nabla\cdot\mathbf{v}_1 + \nabla\cdot(\bar{\rho}_A\,\mathbf{v}_1) = 0 \qquad (18)$$

$$\frac{\epsilon u_1}{\rho_{B_p}} + \frac{k}{\bar{\mu}}\left[1+\frac{\bar{\rho}_A}{B}\right]\frac{\partial u_1}{\partial t} = -\frac{k}{\bar{\mu}}\left[\frac{1}{\rho_{B_p}}\frac{\partial p_1}{\partial x} - g\frac{\rho_{A_1}}{B}\right] - \frac{\mu_1}{\bar{\mu}}\frac{\epsilon}{\rho_{B_p}}v_0 \qquad (19)$$

$$\frac{\epsilon v_1}{\rho_{B_p}} + \frac{k}{\bar{\mu}}\left[1+\frac{\bar{\rho}_A}{B}\right]\frac{\partial v_1}{\partial t} = -\frac{k}{\bar{\mu}\,\rho_{B_p}}\frac{\partial p_1}{\partial y} \qquad (20)$$

$$\frac{\epsilon w_1}{\rho_{B_p}} + \frac{k}{\bar{\mu}}\left[1+\frac{\bar{\rho}_A}{B}\right]\frac{\partial w_1}{\partial t} = -\frac{k}{\bar{\mu}\,\rho_{B_p}}\frac{\partial p_1}{\partial z} \qquad (21)$$

$$\frac{\partial \rho_{A_1}}{\partial t} + \nabla\cdot(\bar{\rho}_A\mathbf{v}_1) = \frac{\kappa B}{B+\bar{\rho}_A}\left[\nabla^2\rho_{A_1} - \frac{2}{B+\bar{\rho}_A}(\nabla\bar{\rho}_A)\cdot(\nabla\rho_{A_1}) + \frac{\rho_{A_1}}{(B+\bar{\rho}_A)^2}(\nabla\bar{\rho}_A)^2\right] \qquad (22)$$

One wishes to solve these equations for a flow system sufficiently general to illustrate the importance of a number of stability criteria, but also sufficiently simple to be mathematically tractable. Hence the following physically unrealistic but mathematically convenient steady displacement is chosen. Consider the infinite porous medium shown in Figure 1. Imagine a plane at $x = 0$ moving at velocity v_0. Above this plane pure component A is maintained. Below a plane located at $x = h$ pure B is maintained. It is important to remember that one is interested in distinguishing stable displacements from unstable displacements within the mixing zone between $x = 0$ and $x = h$. No attempt has been made to describe the progress of macroscopic fingering which develops after an initial instability.

A steady state solution to Equation (13) is obtained satisfying the boundary conditions

$$\bar{\rho}_A = \rho_{A_p};\ x \leqq 0 \qquad (23)$$

$$\bar{\rho}_A = 0;\ x \geqq h \qquad (24)$$

These lead to the solution (6)

$$\bar{\rho}_A = B\left[(J)^{\frac{x}{h}-1} - 1\right] \qquad (25)$$

from which $\bar{\mu}$ is obtained through the relation

$$\bar{\mu} = \mu_{B_p}\,exp\,(L\bar{\rho}_A) \qquad (26)$$

One now focuses his attention on one Fourier component of the perturbations, which one writes in the form

$$(u_1, v_1, w_1) = [U(x), V(x), W(x)]\ \exp\,[i(ly + mz) + \sigma t] \qquad (27)$$

$$\rho_{A_1} = Q(x)\,\exp\,[i(ly + mz) + \sigma t] \qquad (28)$$

$$\mu_1 = \bar{\mu}\,L\,\rho_{A_1} \qquad (29)$$

The goal is to derive an ordinary differential equation for one of the pre-exponential functions, say $Q(x)$, in the perturbations given above. A partial differential equation from which the pressure has been eliminated is obtained by taking ∇_1^2, $\frac{\partial^2}{\partial x\,\partial y}$, and $\frac{\partial^2}{\partial x\,\partial z}$ of Equations (19), (20), and (21), respectively, and combining the results:

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(v_1\bar{\mu}) + \frac{\partial}{\partial z}(w_1\bar{\mu})\right] + \frac{k\rho_{B_p}}{\epsilon}\frac{\partial}{\partial x}\left\{\left[1+\frac{\bar{\rho}_A}{B}\right]\frac{\partial}{\partial t}\left(\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z}\right)\right\}$$
$$= \nabla_1^2(u_1\bar{\mu}) + \frac{k\rho_{B_p}}{\epsilon}\left[1+\frac{\bar{\rho}_A}{B}\right]\nabla_1^2\frac{\partial u_1}{\partial t} - \frac{g\rho_{B_p}k}{\epsilon B}\nabla_1^2\rho_{A_1} + v_0\,\nabla_1^2\mu_1 \qquad (30)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The perturbation functions of Equation (27) are related to $Q(x)$ as follows. From the continuity Equation (18) and the diffusion Equation (22) one finds

$$U = -\frac{\sigma Q}{\bar{\rho}_A'} + \frac{\kappa}{\bar{\rho}_A'}\left\{Q'' - (l^2 + m^2)\,Q + \frac{(\bar{\rho}_A')^2}{(B+\bar{\rho}_A)^2}Q - \frac{2\bar{\rho}_A'}{B+\bar{\rho}_A}Q'\right\} \qquad (31)$$

The continuity equation can be written

$$\sigma Q + (\bar{\rho}_A + B)\,[U' + ilV + imW] + U\bar{\rho}_A' = 0 \quad (32)$$

Cross differentiation of Equations (20) and (21) yields

$$m\,V = l\,W \qquad (33)$$

Thus from Equations (31), (32), and (33) one can express U, V, W, and their derivatives in terms of Q and its derivatives. These substitutions are now made in Equation (30) which reduces, after much algebra and utilization of Equation (25), to the desired ordinary differential equation

$$AP^{IV} + CP''' + DP'' + EP' + FP = 0 \qquad (34)$$

where one has made the convenient changes in variables

$$X = x/h$$
$$P(X) = Q(x)$$

and

$$A = \left[1 + \frac{\sigma k\rho_{A_p}}{\bar{\mu}\,\epsilon}J^X\right]$$

$$C = \ln J\,[\alpha J^X - 3] - \frac{2\,\sigma k\rho_{A_p}}{\bar{\mu}\,\epsilon}(\ln J)\,J^X$$

$$D = (\ln J)^2\,[3 - 2\alpha J^X] - 2b^2 + \frac{\sigma k\rho_{A_p}J^X}{\bar{\mu}\,\epsilon}\left[(\ln J)^2 - 2b^2 - \frac{\sigma h^2}{\kappa}\right] - \frac{\sigma h^2}{\kappa}$$

$$E = (\ln J)^3\,[\alpha J^X - 1] + (\ln J)\,b^2\,[3 - \alpha J^X] + \frac{\sigma k\rho_{A_p}}{\bar{\mu}\,\epsilon}J^X(\ln J)\left[2b^2 + \frac{\sigma h^2}{\kappa}\right] + \frac{\sigma h^2}{\kappa}(\ln J)\,[2 - \alpha J^X]$$

$$F = b^2\left\{b^2 - (\ln J)^2 - \alpha\beta(\ln J)\,J^X\right.$$

$$- \lambda \exp [\alpha J (1 - J^{X-1})] J^X + \frac{\rho_{A_p} J^X k \sigma}{\mu \epsilon}$$

$$\left[b^2 - (\ln J)^2 + \frac{\sigma h^2}{\kappa} \right] \Bigg\} + \frac{\sigma h^2}{\kappa} [(\ln J)^2 (\alpha J^X - 1) + b^2]$$

Dimensionless groups appearing in Equation (34) are

$$b^2 = h^2 (l^2 + m^2)$$

$$\alpha = \frac{L \rho_{A_p}^2}{\rho_{A_p} - \rho_{B_p}}$$

$$\beta = \frac{v_0 h}{\kappa}$$

$$\lambda = \frac{- \rho_{A_p} k g h}{\kappa \epsilon \mu_{B_p}} (\ln J)$$

A displacement process is considered stable if small perturbations to the steady state solution, Equation (25), are damped in time, while an unstable displacement is indicated by perturbations which grow in time. One is interested in determining those conditions which divide a stable displacement from an unstable one. From the perturbation Equations (27) through (29) one sees that the displacement is stable whenever the real part of σ, which in general may be complex, is negative and unstable when the real part of σ is positive. The condition of interest is that of marginal stability, which occurs when σ_R, the real part of σ, is zero. If the imaginary part of σ is also zero, one has a condition of stationary stability dividing perturbations which are damped exponentially from those which grow exponentially. On the other hand, $\sigma_R = 0$, $\sigma_i \neq 0$ corresponds to a condition of oscillatory stability, which divides perturbations which grow or are damped in an oscillatory manner (4). For the case of small density differences, $(\rho_{A_p} - \rho_{B_p}) << \rho_{A_p}$, Lapwood (7) has shown that the condition of exchange of stabilities is valid; namely, that for all physically realizable systems of interest, $\sigma_i = 0$, and therefore one can state that the only condition of marginal stability occurs when $\sigma = \sigma_R = \sigma_i = 0$. In the more complicated case treated here, the validity of the principle of exchange of stabilities has not been proved. Nevertheless, one shall consider only the conditions for stationary stability. Instabilities observed in the laboratory have been of the nonoscillatory type. By requiring $\sigma = 0$ one can simplify Equation (34) and obtain the equation which describes marginal stability of the stationary type:

$$P^{IV} + \ln J [\alpha J^X - 3] P''' + \{(\ln J)^2 [3 - 2\alpha J^X] - 2b^2\} P''$$
$$+ \{(\ln J)^2 [\alpha J^X - 1] + b^2 [3 - \alpha J^X]\} (\ln J) P'$$
$$+ \{b^2 - (\ln J)^2 - \alpha\beta (\ln J) J^X - J^X \lambda$$
$$\exp [\alpha J (1 - J^{X-1})]\} b^2 P = 0 \quad (35)$$

The dimensionless group λ is analogous to the classic Rayleigh number. In the case where $(\rho_{A_p} - \rho_{B_p}) << \rho_{A_p}$, λ becomes numerically equivalent to the dimensionless group used by Lapwood in the simple thermal convection problem. It is a measure of the convective influence of buoyancy relative to the damping effects of diffusion and viscous drag. In the current problem several additional dimensionless groups appear. The group α reflects sensitivity of viscosity to changes in concentration, as shown in Equation (26). The parameter β is a measure of the effect of mean displacement velocity. In most previous analyses this effect has not been evident because of the simplified nature of the problem studied.

Boundary Conditions. From the conditions under which the steady state solution (25) was obtained one can readily specify four boundary conditions on P. It is required that all perturbations cease at $X = 0$ and $X = 1$. From this requirement, and from Equation (9)

$$P(0) = 0 \quad (36)$$

$$P''(0) - 2(\ln J) P'(0) = 0 \quad (37)$$

$$P(1) = 0 \quad (38)$$

$$P''(1) - 2(\ln J) P'(1) = 0 \quad (39)$$

METHOD OF COMPUTATION

Equation (35) with the associated boundary conditions (36) through (39) constitutes a classic eigenvalue problem. For a given set of parameters J, α, β, and b^2 one wishes to find the minimum value of λ which will admit a solution to (35) satisfying the boundary conditions. The set of parameters J, α, β, b^2, and λ then defines a condition of stationary stability. For a given J, α, and β the critical value of λ, denoted by λ_c, is found by varying b^2 to find the value which gives the smallest value of λ. Since a disturbance may contain any combination of Fourier components, it is presumed that the smallest value of λ so found corresponds to the critical condition of stationary stability for the displacement. By carrying out this type of calculation for several combinations of J, α, and β one obtains a four-dimensional map of stationary stability.

The computations were performed numerically with a scheme described by Fox (5). The differential Equation (35) was written as a difference equation with suitable modifications being made at each boundary to incorporate the boundary conditions. In most cases forty-one intervals between $X = 0$ and $X = 1$ were sufficient. Difference corrections were included in the manner described by Fox. For a given choice of J, α, β, and b^2 the result is a matrix equation of the form

$$[\mathfrak{A} - I \lambda][\mathfrak{P}] = 0 \quad (40)$$

To find the smallest eigenvalue λ the method of matrix iteration (5) is applied to

$$[\mathfrak{A}^{-1} - I \lambda^{-1}][\mathfrak{P}] = 0 \quad (41)$$

The computer was programed to search for the value of b^2 which resulted in a minimum value of λ for given J, α, and β.

RESULTS AND DISCUSSION

The values of dimensionless wave number squared (b^2) at which λ_c was obtained are presented in Table 1*. For $-10 \leqq \alpha \leqq 10$, $0.60 < J < 1.0$, and $\beta = 0$ or 10, the value of b^2 at λ_c is relatively constant, ranging between 9.8 and 16. However, as the density ratio J is reduced, b^2 becomes increasingly sensitive to changes in α. For example, for $J = 0.199$, $\beta = 0$, and $\alpha = -10$, one finds $\lambda_c = 1.146$ and $b^2 = 98.95$.

The effect of the various pertinent dimensionless groups J, α, and β on λ_c is shown in Figures 2 through 4. Figure 2*a* illustrates the result which follows from the Boussinesq approximation, namely, that differences in density are important only in the buoyancy term (the term containing λ). In this case the results are identical to the heat transfer problem with insulated boundaries which has been treated by Lapwood (7), and the results compare favorably with his solution which yields $\lambda_c = 4\pi^2$, $b^2 = \pi^2$. It is apparent from Equation (35) that in this limiting case viscosity variation and bulk flow do not affect the value of λ_c.

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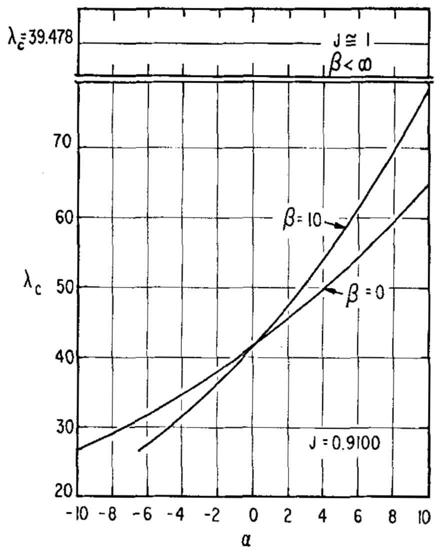

Fig. 2a. Critical values of λ with Boussinesq approximation. 2b. Critical values of λ for J = 0.9100.

As the density ratio J decreases from unity, the parameters α and β [the latter appearing in Equation (35) as the product $\alpha\beta$] must be considered. The mean motion, which appears in β, acts as a destabilizing influence if the viscosity of the displacing fluid is less than that of the displaced fluid. If the viscosity of the displacing fluid exceeds that of the displaced fluid, the mean motion acts as a stabilizing influence. This effect can be traced to the coupling between mean motion and viscosity perturbations shown in Equation (19). As would be expected, the value of β becomes increasingly influential as J is decreased from unity. For example, at $\alpha = -5$, λ_c is reduced by 25% for $J = 0.8$ as β increases from 0 to 10. However, at $J = 0.6$, the corresponding reduction is 47%. The increasing importance of α as J is decreased from unity is evident in Figures 2, 3, and 4, especially when it is noted that in the latter two figures λ_c is plotted on a logarithmic scale. Negative values of α correspond to a displacing fluid which has a lower viscosity than the fluid being displaced. Physically, one would expect the ratio (viscosity of displacing fluid/viscosity of displaced fluid) to have a stabilizing or destabilizing effect depending upon whether the ratio is greater or less than unity, respectively. The figures show that such is indeed the case. It is important to note that the effect of viscosity variation, though qualitatively dependent upon the ratio of pure-component viscosities, requires a means of expression of the local viscosity throughout the area of interpenetration of the two fluids. This fact points out limitations inherent in simplified expressions which are sometimes used to predict stability limits and which express the effect of viscosity in terms of the pure-component ratio.

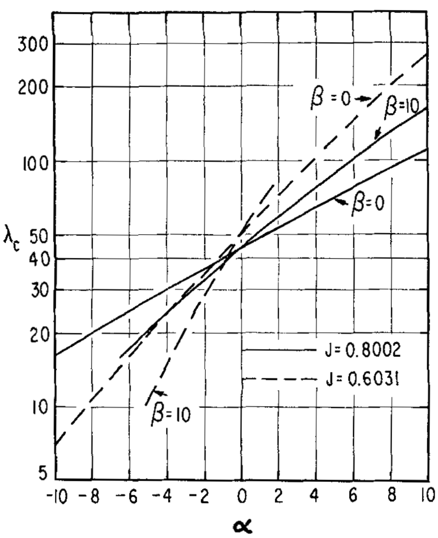

Fig. 3. Critical values of λ for J = 0.8002 and J = 0.6031.

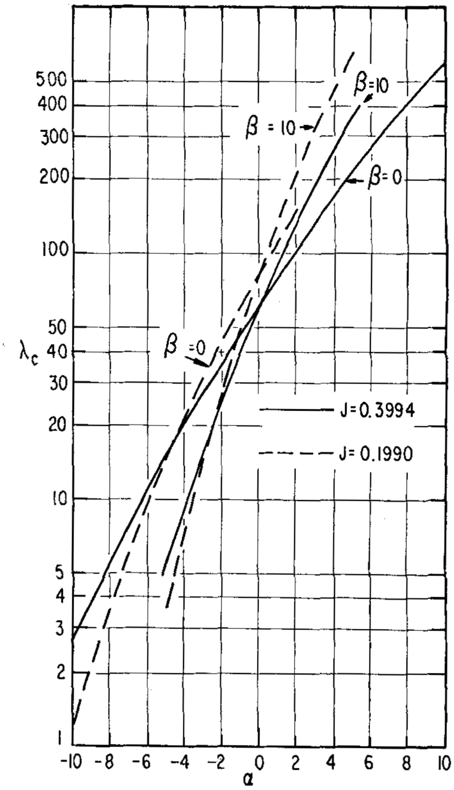

Fig. 4. Critical values of λ for J = 0.3994 and J = 0.1990.

It is of interest to compare the form of λ used here with that used by Lapwood [Equation (1)]. In the present case

$$\lambda = \frac{-\rho_{A_p} kgh}{\kappa\, \epsilon\, \mu_{B_p}} (\ln J) \tag{42}$$

In the Boussinesq approximation, where $J \cong 1$, one can expand $\ln J$ and easily show

$$\lambda = \frac{kg(\Delta\rho)h}{\kappa\, \epsilon\, \mu_{B_p}} \tag{43}$$

where $\Delta\rho = (\rho_{A_p} - \rho_{B_p})$. Equation (43) corresponds, for a linear density gradient, to the form used by Lapwood and Wooding.

Also of interest is a study of λ_c as a function of J for $\alpha = \beta = 0$. It is seen from Table 1* that λ_c increases as J decreases. One may surmise from this that, contrary to physical intuition, decreasing J has a stabilizing effect. However, the explicit effect of J in λ_c may be removed by computing $|\lambda_c/(\ln J)|$. This is a monotonic function, decreasing with decreasing J. For nonzero values of α and/or β the situation is more complicated.

Since the condition of marginal stability is determined by α, β, and λ_c for a given value of J, one can imagine sufficiently adverse values of α and β such that λ_c has a negative value. This was observed in some trial computations.

* See footnote on p. 103.

The importance of the supplementary parameters α and β is easily seen by considering a sample displacement. Suppose

$$\rho_{A_p} = 1.0 \text{ g./cc.}$$

$$\rho_{B_p} = 0.8 \text{ g./cc.}$$

$$\mu_{A_p} = 1 \text{ centipoise}$$

$$\mu_{B_p} = 10 \text{ centipoise}$$

$$v_o = 1 \text{ ft./day}$$

$$\kappa = 1 \times 10^{-5} \text{ sq. cm./sec.}$$

$$h = 100 \text{ ft.}$$

For these conditions one obtains $\alpha = -11.5$, $\beta = 1.08 \times 10^5$. Consequently, λ_c will be quite different from 39.5 given by the Boussinesq approximation or 44.1 for $J = 0.8$, but $\alpha = \beta = 0$.

CONCLUSIONS

1. For the miscible displacement considered in this paper, a condition of marginal stability is determined by the four parameters α, β, J, and λ_c.
2. The ratio μ_{A_p}/μ_{B_p} has a stabilizing or destabilizing effect for values greater or lesser than 1, respectively. However, it is important to have a realistic expression relating local viscosity to composition. This expression is contained in the parameter α.
3. A nonzero (positive) displacement velocity has a stabilizing or destabilizing effect for positive or negative values of α, respectively.
4. The importance of nonzero α and β is accentuated as the density ratio J decreases from unity.

ACKNOWLEDGMENT

Work on this problem was initiated while the author was serving as a summer consultant to the Production Research Division of the Humble Oil and Refining Company, Houston, Texas. Dr. H. H. Rachford, Jr., made several helpful suggestions. Partial support from the Research Corporation is also gratefully acknowledged.

NOTATION

$\mathfrak{A}$ = matrix coefficient of difference Equation (40)
B = $\rho_{A_p}\rho_{B_p}/(\rho_{A_p} - \rho_{B_p})$
b^2 = $(l^2 + m^2)/h^2$
D/Dt = substantial derivative
d = characteristic length
g = acceleration due to gravity
h = length of diffusion zone, Figure 1
I = identity matrix
i = $\sqrt{-1}$
J = ρ_{B_p}/ρ_{A_p}
k = permeability
L = constant relating viscosity to concentration, defined by Equation (26)
l = wave number defined by Equation (27)
m = wave number defined by Equation (27)
$P(X)$ = $Q(x)$
$\mathfrak{P}$ = matrix of dependent variable P
p = pressure
Q = concentration perturbation function defined by Equation (28)
Q_v = volumetric flow rate
q = superficial velocity
R = $\dfrac{-g\,d^4}{\kappa\,\mu}\dfrac{d\bar{\rho}}{dx}$
t = time
U,V,W = velocity perturbation functions defined by Equation (27)
u,v,w = components of $\mathbf{v}$ in x, y and z directions, respectively
$\mathbf{v}$ = vector pore velocity defined by Equation (3)
v_0 = velocity in x direction of moving coordinate system
v_A or v_B = pore velocity of component A or B, respectively, defined by Equation (7)
X = x/h
x = distance coordinate taken in the direction of gravity

Greek Letters

α = $L\rho_{A_p}^2/(\rho_{A_p} - \rho_{B_p})$
β = $v_0 h/\kappa$
ϵ = void fraction
κ = molecular diffusivity or effective dispersion coefficient
κ_T = thermal diffusivity
λ = $-\dfrac{\rho_{A_p}\,kgh}{\kappa\,\epsilon\,\mu_{B_p}}(\ln J)$
λ_c = critical value of λ
μ = viscosity
μ_{A_p} or μ_{B_p} = pure-component viscosity
ρ = density
ρ_A or ρ_B = local density of component A or B, respectively
ρ_{A_p} or ρ_{B_p} = pure-component density of A or B, respectively
σ = growth parameter defined by Equation (27)

Subscript

1 = perturbation value

Superscripts

— = steady state value
′ = fixed coordinate system or differentiation

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Manuscript received January 13, 1964; revision received June 12, 1964; paper accepted June 15, 1964.